Final Exam

Name: ______________________________

Sign below to affirm that you followed the Duke University community standard on this exam.

- I will not lie, cheat, or steal in my academic endeavors;
- I will conduct myself honorably in all my endeavors; and
- I will act if the standard is compromised.

Signature: ______________________________

Instructions:

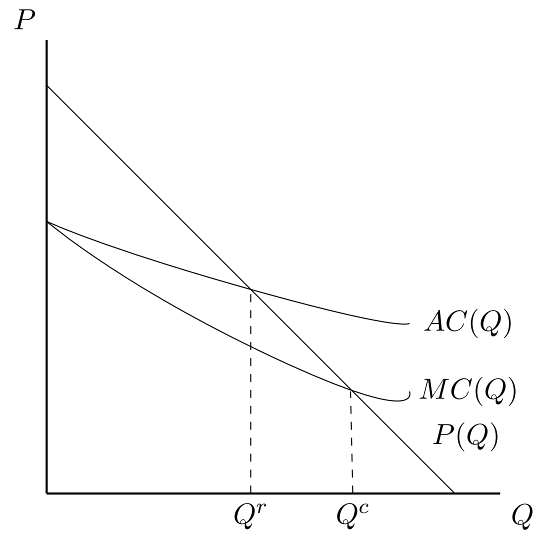
- Questions are not ordered from easiest to hardest. It might be wise to find the problems you consider easiest and do them first.
- If you can't complete a question because it relies on an earlier answer, describe how you would have done it for partial credit.
- Show all work.
- You have 180 minutes to complete the exam.

Question:	1	2	3	4	5	6	Total
Points:	15	15	20	22	20	16	108
Score:							

1. Short answers: briefly justify your response.

   (a) (5 points) Graph marginal cost, average cost, and demand for one type of natural monopoly. Label the output levels that a regulator might allow. What is one problem with each output level?

   Solution: A graph for either type of natural monopoly (IRTS or high fixed costs) is acceptable. The only requirement is that $AC(Q)$ decline. The two output levels are Q^r, where the natural monopoly breaks even, and Q^c, the socially efficient level of output.



   The problem with Q^r is that it creates deadweight loss. The problem with Q^c is that natural monopolist makes a loss and must be reimbursed by the government. No more detail was required, but the practical difficulty is that it is hard to make the monopolist truthfully report its loss.

   (b) (5 points) Suppose that a monopolist with $C(Q) = 2Q$ serves a market with demand $P(Q) = 10 - Q$. If the monopolist uses perfect price discrimination, what are consumer surplus, profit, and total surplus? Does the outcome maximize total surplus?

   Solution: The monopolist produces the socially optimal quantity Q^c such that $P(Q^c) = MC(Q^c)$, in this case $Q^c = 10 - MC = 8$. Total surplus is the area between the demand curve and marginal cost curve, $\frac{1}{2}Q^c(P(0) - P(Q^c)) = \frac{1}{2}(8)(10-2) = \frac{8^2}{2} = 32$. Because every consumer pays exactly his willingness to pay, consumer surplus is zero and profit equals total surplus.

   (c) (5 points) Two duopolists compete in a market. They have the same best-response functions: $BR_1(q_2) = 36 - \frac{1}{2}q_2$ and $BR_2(q_1) = 36 - \frac{1}{2}q_1$. What are Nash equilibrium levels of output?

   Solution: The duopolists need a simultaneous best-response,

$$q_1^* = BR_1(q_2^*) = BR_1(BR_2(q_1^*))$$

$$
\begin{aligned}
q_1^* &= 36 - \frac{1}{2}\left(36 - \frac{1}{2}q_1\right) \\
&= 18 + \frac{1}{4}q_1^* \\
\frac{3}{4}q_1^* &= 18 \\
q_1^* &= 24 \\
q_2^* &= BR_2(16) = 24.
\end{aligned}
$$

2. Indicate whether each statement is *always true* or *can be false*. **Briefly justify your answer.** Correct answers with no justification receive one point.

   (a) (5 points) Consider a game in a Nash equilibrium. If two players simultaneously change their strategies, they cannot improve their payoffs.

   > **Solution:** *False.* In a Nash equilibrium, no player can improve his payoff by changing his strategy *holding all other strategies constant.* However, if two players change strategies at the same time, they could both improve their payoffs. In the prisoner's dilemma, a simultaneous deviation could take players from (R, R) to the better outcome (Q, Q).

   (b) (5 points) A per-unit tax on one good always creates deadweight loss.

   > **Solution:** *False.* When there is a negative externality from production of a good, a per-unit tax on that good can raise total surplus.

   (c) (5 points) Naomi is risk-averse and faces some lottery $(x_1, x_2; \alpha, 1-\alpha)$. Insurance company #1 offers actuarially fair insurance. Company #2 offers to trade her lottery for her certainty equivalent, x^{CE}. Naomi should accept company #2's offer.

   > **Solution:** *False.* Naomi is risk-averse, so her certainty equivalent is less than the expected wealth of the lottery. With actuarially fair insurance, she could fully insure and receive her expected wealth with certainty.

3. Juan Martín (JM) buys french fries (x) and ketchup (y) with preferences given by $U(x, y) = xy$. The price of french fries is $p_x = 12$ and the price of ketchup is $p_y = 3$.

(a) (5 points) Find Juan Martín's compensated demands for french fries x and ketchup y as functions of u.

Solution: By the MRS condition,

$$\frac{y}{x} = \frac{12}{3}$$
$$y = 4x.$$

Substituting into the constraint $u = xy$ gives

$$u = (4x)x = 4x^2$$
$$x^c(u) = \frac{1}{2}\sqrt{u}$$
$$y^c(u) = 2\sqrt{u}.$$

(b) (4 points) Find Juan Martín's expenditure function $e(u)$.

Solution:

$$e(u) = p_x x^c(u) + p_y y^c(u) = 12(\frac{1}{2}\sqrt{u}) + 3(2\sqrt{u}) = 12\sqrt{u}.$$

(c) (2 points) Suppose Juan Martín spends all of his income and achieves utility 16. What are his consumption bundle and income?

Solution: From compensated demands, Juan Martín's consumption bundle must be $(x^c(16), y^c(16)) = (2, 8)$. The expenditure of that bundle (and hence income) is $12\sqrt{16} = 12 \cdot 4 = 48$.

In response to a shortage of tomatoes, the government of Argentina freezes the price of ketchup and begins rationing. The price of ketchup stays the same at $p_y = 3$, but now Juan Martín can buy at most 4 units: $y \leq 4$.

(d) (2 points) If Juan Martín spends all of his income from (c), what is his optimal bundle when $y \leq 4$?

Solution: The new bundle involves buying 4 units of y and spending the rest of his income on x. Juan Martín buys

$$y^* = 4$$
$$x^* = \frac{48 - 4 \cdot 3}{12} = 3.$$

(e) (4 points) How much compensation does Juan Martín need to be as well off as before the shortage?

Solution: Juan Martín must reach the original utility level 16 while only buying 4 units of y. Consumption of x must be

$$4x = 16 \Rightarrow x = 4.$$

The cost of the bundle is $4 \cdot 12 + 4 \cdot 3 = 60$. The compensation Juan Martín requires is the difference in expenditure, $60 - 48 = 12$.

(f) (3 points) Find Juan Martín's expenditure function $e(u)$ after the shortage.

Solution: Expenditure is unchanged if Juan Martín's compensated demand calls for 4 units of y or fewer. This utility level solves $y^c(\bar{u}) = 4$, so $\bar{u} = 4$. Beyond $\bar{u}$, any utility u must be reached by buying 4 units of y and the amount of x solving $xy = 4x = u$, or $x = \frac{u}{4}$. Expenditure is then

$$e(u) = \begin{cases} 12\sqrt{u} & \text{if } u \leq 4 \\ 3u + 12 & \text{if } u > 4. \end{cases}$$

4. Four consumers have preferences $U_i(x,y) = y - \frac{1}{2}(7-x)^2$ over money y and cake x. Money (y) is a numéraire with $p_y = 1$, but cake (x) is produced by a (simple) monopoly with $C(Q) = 3Q$ that maximizes profit and charges p per unit. The four consumers, however, own the monopoly and receive its profit as income. Each consumer's income is $20.5 + \frac{\pi}{4}$, where π is the profit earned by the monopoly. (You may ignore corner solutions throughout this question.)

(a) (5 points) Calculate ordinary demands for x and y as a function of p and π.

Solution:

$$\begin{aligned}
\mathcal{L} &= y - \frac{1}{2}(7-x)^2 + \lambda(20.5 + \frac{\pi}{4} - px - y) \\
\frac{\partial \mathcal{L}}{\partial y} &= 1 - \lambda = 0 \\
\frac{\partial \mathcal{L}}{\partial x} &= (7-x) - \lambda p = 0 \\
x^*(p) &= 7 - p \\
y^*(p, 20.5, \pi) &= 20.5 + \frac{\pi}{4} - p(7-p).
\end{aligned}$$

(b) (2 points) Find market demand for cake x, $Q_D(p)$. (Inverse demand $p(Q)$ is also acceptable.)

Solution: All four consumers demand $x_i^*(p) = 7 - p$. Then $Q_D(p) = 28 - 4p$, or $p(Q) = 7 - \frac{1}{4}Q$.

(c) (4 points) What price and quantity does the monopoly set?

Solution: The monopoly maximizes profit,

$$\begin{aligned}
&\max_p p(28-4p) - 3(28-4p) \\
\frac{\partial}{\partial p} &= 28 - 8p + 12 = 0 \\
8p^m &= 40 \\
p^m &= 5 \\
Q^m &= 28 - 20 = 8.
\end{aligned}$$

(d) (2 points) Calculate the monopoly's profit. How much income does each consumer get from the firm?

Solution:

$$\pi^m = 5(8) - 3(8) = 16 \Rightarrow \frac{\pi}{4} = 4.$$

One of the consumers proposes that the monopoly produce the socially efficient quantity. Another consumer points out that the consumers receive less profit under this proposal.

(e) (3 points) Without any calculations, explain whether the proposal will benefit the consumers.

Solution: It benefits the consumers because the socially efficient quantity increases total surplus, more than compensating consumers for the loss of profit.

(f) (3 points) Calculate the level of output Q^c and price p^c in the market for cake (x) under the proposal. What is profit?

Solution: The monopolist produces the quantity Q^c such that

$$\begin{aligned} MC(Q^c) &= P(Q^c) \\ 3 &= 7 - \frac{1}{4}Q^c \\ \frac{1}{4}Q^c &= 7 - 3 = 4 \\ Q^c &= 16 \\ p^c &= 7 - \frac{1}{4}(16) = 3. \end{aligned}$$

Profit is $\pi = 3 \cdot 16 - 3 \cdot 16 = 0$.

(g) (3 points) Is ΔCS in the market for cake (x) a good measure of the change in welfare from the proposal? (You do not need to calculate ΔCS.)

Solution: No. ΔCS describes the benefits of increased consumption of x at the new price, but it ignores that the consumers are buying less of good y after the decrease in profit.

5. A firm in the market for ice cream has production function $f(l,k) = 2\sqrt{lk}$. The firm is in the short run with $k_0 = 1$ units of capital. The wage is $w = 2$ and the rental rate is $v = 12$. The market is perfectly competitive, so the firm charges the market price p.

   (a) (4 points) Find the firm's short-run cost function.

   Solution: To produce output q, the firm needs the level of labor $l(q)$ solving $q = 2\sqrt{k_0 l(q)}$. This gives $l(q) = \frac{q^2}{4k_0} = \frac{q^2}{4}$. The cost function is

   $$SC(q) = wl(q) + vk_0 = \frac{q^2}{2} + 12.$$

   Changes in the weather affect demand for ice cream. With probability $\frac{1}{2}$, it will be hot and the firm can charge $p_H = 10$ for each unit, but with probability $\frac{1}{2}$ it will be cold and the firm can only charge $p_L = 6$. The firm must choose its output q *before* learning about the price. It has Bernoulli utility over profit $u(\pi) = \pi$.

   (b) (2 points) What is the firm's profit $\pi_L(q)$ from producing q when the price is low? What is the profit $\pi_H(q)$ from producing q when the price is high? (Your answer should only depend on q and constants.)

   Solution:

   $$\pi_L(q) = p_L q - \frac{q^2}{2} - 12 = 6q - \frac{q^2}{2} - 12$$
   $$\pi_H(q) = p_H q - \frac{q^2}{2} - 12 = 10q - \frac{q^2}{2} - 12.$$

   (c) (3 points) What lottery does the firm face when it produces q units?

   Solution: The firm earns $\pi_L(q)$ with probability $\frac{1}{2}$ and $\pi_H(q)$ with probability $\frac{1}{2}$. The lottery is

   $$(\pi_L(q), \pi_H(q); \frac{1}{2}, \frac{1}{2}) = (6q - \frac{q^2}{2} - 12, 10q - \frac{q^2}{2} - 12; \frac{1}{2}, \frac{1}{2}).$$

(d) (4 points) How much ice cream q does the firm produce? What is its expected profit?

Solution: The firm chooses the q maximizing its expected utility from the lottery,

$$\begin{aligned}
\text{EU}(\pi_L(q), \pi_H(q); \frac{1}{2}, \frac{1}{2}) &= \frac{1}{2}\left(6q - \frac{1}{2}q^2 - 12\right) + \frac{1}{2}\left(10q - \frac{1}{2}q^2 - 12\right) \\
&= 8q - \frac{1}{2}q^2 - 12 \\
\frac{\partial}{\partial q} &= 8 - q^* = 0 \\
q^* &= 8.
\end{aligned}$$

Expected profit is $8^2 - \frac{1}{2}8^2 - 12 = 64 - 32 - 12 = 20$.

(e) (3 points) What would the firm's quantity and profit be if it knew it would be cold (price p_L)? What if it knew it would be hot (price p_H)?

Solution: There is no uncertainty, so the firm can select its optimal quantity for each price.

$$\begin{aligned}
\pi_L(q) &= 6q - \frac{1}{2}q^2 - 12 \\
\frac{\partial}{\partial q} &= 6 - q = 0 \\
q_L^* &= 6 \\
\pi_L(6) &= 36 - 18 - 12 = 6.
\end{aligned}$$

$$\begin{aligned}
\pi_H(q) &= 10q - \frac{1}{2}q^2 - 12 \\
\frac{\partial}{\partial q} &= 10 - q = 0 \\
q^* &= 10 \\
\pi_H(10) &= 100 - 50 - 12 = 38.
\end{aligned}$$

(f) (4 points) Suppose the firm can hire a forecaster that tells the firm whether it will be hot or cold *before* the firm chooses its quantity. However, the probabilities are unchanged: the forecaster announces $p_L = 6$ half the time and $p_H = 10$ half the time. What is the most the firm would pay the forecaster?

> **Solution:** The firm will pay the forecaster until its profit with the forecast equals its profit without. Expected profit without the forecast is 20. Profit with the forecast is
>
> $$\frac{1}{2}\pi_L(6) + \frac{1}{2}\pi_H(10) = \frac{1}{2}(6) + \frac{1}{2}38 = 3 + 19 = 22.$$
>
> The firm is willing to pay up to $22 - 20 = 2$ units for the forecast.

6. Consider the matrix game below. Assume that all players are risk-neutral: $u(x) = x$.

Player 2 (columns); Player 1 (rows)

	L	C	R
T	$(3,3)$	$(1,0)$	$(2,2)$
M	$(0,1)$	$(3,0)$	$(1,1)$
B	$(2,2)$	$(4,4)$	$(5,1)$

(a) (5 points) Find all of the pure-strategy Nash equilibria of the game.

> **Solution:** The two NE are (T, L) and (B, C).

(b) (3 points) Are any of the pure-strategy Nash equilibria Pareto efficient? Explain briefly.

> **Solution:** (B, C) is Pareto efficient but (T, L) is not. At (T, L), both players are better off if they move to (B, C). At (B, C), the only way to make P1 better off is to move to (B, R), but this harms P2.

(c) (2 points) Are any other pure-strategy outcomes Pareto efficient? Explain briefly.

> **Solution:** Yes: (B, R) is Pareto efficient even though it is not a Nash equilibrium. At (B, R), any movement makes player 1 worse off, so no Pareto improvement is possible.

(d) (6 points) Suppose player 1 mixes over T and B and player 2 mixes over L and C. (Player 1 never plays M and player 2 never plays R.) Player 1 plays T with probability p_T and B with probability $1 - p_T$. Player 2 plays L with probability p_L and C with probability $1 - p_L$. Find a mixed-strategy Nash equilibrium.

Solution: Player 1's mixing must make player 2's payoffs from L and C equal,

$$\begin{aligned}
\text{Exp. Payoff from } L &= \text{Exp. Payoff from } C \\
p_T(3) + (1 - p_T)(2) &= p_T(0) + (1 - p_T)(4) \\
p_T + 2 &= 4 - 4p_T \\
5p_T &= 2 \\
p_T^* &= \frac{2}{5}.
\end{aligned}$$

Similarly, player 2's mixing must make player 1's payoffs from T and B equal,

$$\begin{aligned}
\text{Exp. Payoff from } T &= \text{Exp. Payoff from } B \\
p_L(3) + (1 - p_L)(1) &= p_L(2) + (1 - p_L)(4) \\
2p_L + 1 &= 4 - 2p_L \\
4p_L &= 3 \\
p_L^* &= \frac{3}{4}.
\end{aligned}$$

The Nash equilibrium is for P1 to play T w.p. $\frac{2}{5}$ and B w.p. $\frac{3}{5}$ and for P2 to play L w.p. $\frac{3}{4}$ and C w.p. $\frac{1}{4}$.